TRUE SENTENCES AND CORRECT INFERENCES

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Abstract. Deduction Theorem in Mathematical Logic shows a relation between true sentences and correct inferences. In this paper, we shall prove a theorem which shows a relation between Godel Completeness Theorem and Deduction Theorem. This fact is called here "Gentzen-Hilbert Theorem". Moreover, we shall consider this theorem in more general settings.

Logic is a study of correct inferences. But, sometimes we use this word to denote correct inferences themselves. So, it is possible to state that logic is a study of logic. In this paper, we shall use this word to denote these two meanings as far as any confusion does not occur.

Correctness of inferences depends on truth values of sentences in their premises and conclusions. Truth values of sentences depend on meanings of words in them. Meanings of words depend on persons who use them.

Therefore, correctness of inferences and truth values of sentences depend on persons who use them. But, there are sentences which are true for everybody and inferences which are correct for everybody. Such sentences are called here "logically true sentences" and such inferences are called "logically correct inferences" .

Mathematical Logic succeeded in characterizing the set of logically true sentences and the set of logically correct inferences by Godel Completeness Theorem.

But, almost all inferences used in our ordinary life are not logically correct. So, it is desirable to study logic used in our ordinary life, which is called "local logic".

Our purpose of this paper is to investigate local logic. In order to do so, it is necessary to consider relations between true sentences and correct inferences.

For each inference d from $F_1, F_2, \ldots, F_n$ to F, the sentence $\neg F_1 \vee \neg F_2 \vee \ldots \vee \neg F_n \vee F$ is called "kernel" of d and denoted by Ker(d).

Here, we consider the following three conditions on d:

(I) *d is logically correct.*

(II) *If all premises of d are true for p,*
then the conclusion of d is always true for p, for each person p.

(III) *Ker(d) is logically true.*

Usually, (II) is a definition of (I). So, (I) and (II) are equivalent by definition.

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Key Words and Phrases: Deduction Theorem, Completeness Theorem

(Cf.[2] for definitions of correct inferences.)

On the other hand, the statement "(I) and (III) are equivalent", i.e.

"d is logically correct if and only if Ker(d) is logically true"

is called "Deduction Theorem" and a starting point for investigations of relations between true sentences and correct inferences.

Note here that (II) and (III) are also equivalent. Discovery of this fact is a key contribution of Mathematical Logic.

Let S be a society. Then, sentences which are true for everybody in S are called "S-true" and inferences which satisfy the condition obtained from (II) by replacing "for each person p" by "for each person p in S", are called "S-correct". Then, we have the following Deduction Theorem for S:

"d is S-correct if and only if Ker(d) is S-true"

We would like to obtain a characterization theorem of the set of S-true sentences and a characterization theorem of the set of S-correct inferences. But, these are too difficult to obtain meaningful results.

On the other hand, there are two types of Godel Completeness Theorem. One is a characterization theorem of the set of S-correct inferences by using a Gentzen type deduction system. This theorem is called "Godel Completeness Theorem of Gentzen type". Another is a characterization theorem of the set of S-true sentences by using a Hilbert type deduction system. This theorem is called "Godel Completeness Theorem of Hilbert type". We shall show that these two theorems are equivalent in formal languages. This fact will be called here "Gentzen-Hilbert Theorem".

Here, we would like to consider this theorem in more general settings.

In § 1 and § 2 of this paper, we shall develop a general theory of inferences and axiomatizations. In § 3, we shall introduce valuation frames and define true elements and correct inferences. Furthermore, we shall prove Gentzen-Hilbert Theorems in this section. In § 4, we shall deal with formal languages with valuation frames and Gentzen-Hilbert Theorems in them.

§ 1. Inferences and Derivation Sets.

Let G be a non-empty set called "ground set" here. Then, we define inferences on G as follows:

DEFINITION 1.1 (Inferences on G).

An inference d on G is a pair $<\{x_1,x_2,...,x_n\},x>$ *of a finite subset* $\{x_1,x_2,...,x_n\}$ *of G and an element x of G, where* $x_1,x_2,...,x_n$ *are premises of d and x is the conclusion of d .*

An inference $<\{x_1,x_2,...,x_n\},x>$ *will be denoted by* $[x_1,x_2,...,x_n \rightarrow x]$ *and called*

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"inference from $x_1,x_2,...,x_n$ to x". The set of inferences on G is denoted by ***G****.

For each subset D of G*, we define D-proof figures as follows:

DEFINITION 1.2 (Definition of D-proof Figures).

D-proof figures with root and tops are finite trees defined by induction on construction as follows:

(i) An element e of G is a D-proof figure itself. Also, its root and top are e.

(ii) Assume that $[x_1,x_2,...,x_n \rightarrow x]$ is an inference in D and $H_1,H_2,...,H_n$ are D-proof figures such that the root of H_i is x_i for each i=1,2,...,n. Then, the tree H which is obtained from $H_1,H_2,...,H_n$ by adding x as the new root and connecting x_i with x for each i, is a D-proof figure, whose set of tops is the union set of the sets of tops of $H_1,H_2,...,H_n$.

(iii) Every D-proof figure is defined by (i) and (ii) above.

DEFINITION 1.3 (Derivation from A by D and Derivation Set Der(A,D)).

Let A be a subset of G and let D be a subset of G. Then, an element e of G is called "derivable" from A by D if there is a D-proof figure H such the root of H is e and every top of H belongs to A. Let Der(A,D) be the set of elements which are derivable from A by D.*

For example, let G=**N**, A={0} and D={[n→n+2] ; nε**N**}, then Der(A,D)={2n ; nε**N**}, where **N** is the set of natural numbers.

From Definition 1.2 and 1.3 we have:

FACT 1.1. *Any element of A is derivable from A by D.*

FACT 1.2. *If $[x_1,x_2,...,x_n \rightarrow x]$ is an inference in D and $x_1,x_2,...,x_n$ are all derivable from A by D, then x is also derivable from A by D.*

DEFINITION 1.4 (Definitions of d(A) and D(A)).

Let d be an inference $[x_1,x_2,...,x_n \rightarrow x]$ in G and let A be a subset of G. Then,*

$d(A)=A\cup\{x\}$ if all $x_1,x_2,...,x_n$ belong to A, otherwise d(A)=A ,

D(A) = the union set $\cup\{d(A) ; d\varepsilon D\}$.

DEFINITION 1.5. *Let A be a subset of G and let D be a subset of G*.*
Then, A is closed under D if D(A)=A.

From this definition we have:

FACT 1.3. *If A is closed under D, then Der(A,D)=A.*

Hence, we have;

THEOREM 1.1 (Derivation Principle).

Let X be a subset of G. If A is a subset of X and X is closed under D, then Der(A,D) is a subset X.

(Proof) Assume that A is a subset of X and X is closed under D.

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Then, Der(A,D) is a subset of Der(X,D). Since X is closed under D, Der(X,D)=X by Fact 1.3. Therefore, Der(A,D) is a subset X. (Q.E.D.)

Next, let f be a mapping from a ground set G_1 to another ground set G_2. Then, we can extend this mapping to the mapping from G_1^* to G_2^* by

$$f([x_1,x_2,\ldots,x_n\rightarrow x])=[f(x_1),f(x_2),\ldots,f(x_n)\rightarrow f(x)].$$

Also, for each subset A of G_1, each subset D of G_1^* and each subset B of G_2, we define f[A] ,f[D] and $f^{-1}[B]$ by the followings:

$$f[A]=\{f(x)\varepsilon G_2. ; x\varepsilon A\},\ f[D]=\{f(d)\varepsilon G_2^* ; d\varepsilon D\},\ f^{-1}[B]=\{ x\varepsilon G_1 ; f(x)\varepsilon B\}.$$

From these definitions, we have:

FACT 1.4. *Let f be a mapping from G_1 to G_2. Suppose that A is a subset of G_1 and D is a subset of G_1^*. If x ε Ger(A,D), then f(x) ε Ger(f[A],f[D]), for each x in G_1.*

DEFINITION 1.6 (X-conjugate mapping). *Suppose that A is a subset G_1 and f is a mapping from G_1 to G_2.Then, f is "A-conjugate" if x ε A and f(x)=f(y) imply y ε A, for any x, y in G_1 .*

Let f be a mapping form G_1 to G_2. Now, we consider the following two problems:

PROBLEM 1.1. *For each subset A of G_1 and each subset D of G_1^*, find a subset B of G_2 and a subset E of G_2^* such that f[Der(A,D)]=Der(B,E).*

PROBLEM 1.2. *For each subset B of G_2 and each subset E of G_2^*, find a subset A of G_1 and a subset D of G_1^* such that f^{-1}[Der(B,E)]=Der(A,D).*

As for Problem 1.1 we have:

THEOREM 1.2 (Forward Derivation Theorem).

Let f be a Der(A,D)-conjugate mapping from G_1 to G_2 .

Then, f[Der(A,D)]=Der(f[A],f[D]).

(Proof) Assume that f is a Der(A,D)-conjugate mapping from G_1 to G_2..

In order to prove that f[Der(A,D)]=Der(f[A],f[D]), it is sufficient to show the following (1) and (2):

f[Der(A,D)] is a subset of Der(f[A],f[D]) -----(1)

Der(f[A],f[D])] is a subset of f[Der(A,D)] -----(2)

But, (1) is obvious by Fact 1.4.

To show (2), it is sufficient to show the following (3) and (4) by Derivation Principle.

f[A] is a subset of f[Der(A,D)].-----(3)

f[Der(A,D)] is closed under f[D].----(4)

(3) is obvious, because A is a subset of Der(A,D).

To show (4), let d be an arbitrary inference in f[D]. Then, d can be expressed of the form $d=[y_1,y_2,\ldots,y_n\rightarrow y]=f([x_1,x_2,\ldots,x_n\rightarrow x])$ for some $[x_1,x_2,\ldots,x_n\rightarrow x]$ ε D.

Now we assume that all $y_1,y_2,\ldots,y_n$ belong to f[Der(A,D)] and are going to show that

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y belongs to $f[Der(A,D)]$.

Since all $y_1,y_2,...,y_n$ belong to $f[Der(A,D)]$, there are $z_1,z_2,...,z_n$ in $Der(A,D)$ such that $f(z_1)=y_1,f(z_2)=y_2,...,f(z_n)=y_n$.

Since $f(z_1)=f(x_1),f(z_2)=f(x_2),...,f(z_n)=f(x_n)$ and all $z_1,z_2,...,z_n$ belong to $Der(A,D)$, all $x_1,x_2,...,x_n$ belong to $Der(A,D)$,because f is $Der(A,D)$-conjugate.

Hence, x belongs to $Der(A,D)$ by Fact 1.2.

Therefore, we have that y ($=f(x)$) belongs to $f[Der(A,D)]$.

Since d be an arbitrary inference in $f[D]$, this shows (4).

(Q.E.D.)

REMARK 1.1. Let $G_1=\{a,b,c\}$, $G_2=\{d,e\}$, $A=\{b\}$ and $D=\{[a\to c]\}$.Then, $Der(A,D)=A$.

Define the mapping f from G_1 to G_2 by $f(a)=d,f(b)=d$ and $f(c)=e$.

Then, $f[A]=\{d\},f[D]=\{[d\to e]\},f[Der(A,D)]=\{d\}$ and $Der(f[A],f[D])=\{d,e\}$.

These show that the assumption of Theorem 1.2 is necessary.

As for Problem 1.2, we have:

THEOREM 1.3 (Backward Derivation Theorem). *Suppose that f is a mapping from G_1 onto G_2 and g is a mapping from G_2 to G_1 such that $f(g(y))=y$ for all y in G_2. Then,*

$$f^{-1}[Der(B,E)]=Der(g[B],g[E]\cup\{[g(f(x))\to x]\ ;\ x\varepsilon G_1\})$$

(Proof) Assume that f is a mapping from G_1 onto G_2 and g is a mapping from G_2 to G_1 such that $f(g(y))=y$ for all y in G_2.

In order to prove that $f^{-1}[Der(B,E)]=Der(g[B],g[E]\cup\{[g(f(x))\to x]\ ;\ x\varepsilon G_1\})$, it is sufficient to show the following (5) and (6):

$f^{-1}[Der(B,E)]$ is a subset of $Der(g[B],g[E]\cup\{[g(f(x))\to x]\ ;\ x\varepsilon G_1\})$ -----(5)

$Der(g[B],g[E]\cup\{[g(f(x))\to x]\ ;\ x\varepsilon G_1\})$ is a subset of $f^{-1}[Der(B,E)]$ -----(6)

At first, we shall prove (5).

Let z be an arbitrary element of $f^{-1}[Der(B,E)]$. Then, $f(z)\ \varepsilon\ Der(B,E)$.

Therefore, we have $g(f(z))\varepsilon Der(g[B],g[E])$ by Fact 1.4.

By using $[g(f(z))\to z]$,we have $z\ \varepsilon\ Der(g[B],g[E]\cup\{[g(f(x))\to x]\ ;\ x\varepsilon G_1\})$.

Since z is an arbitrary element of $f^{-1}[Der(B,E)]$, this means that (5) holds.

Next, we shall show (6).

Let z be an arbitrary element of $Der(g[B],g[E]\cup\{[g(f(x))\to x]\ ;\ x\varepsilon G_1\})$.

Then, $f(z)\varepsilon\ Der(f[g[B]],f[g[E]]\cup\{[f(g(f(x)))\to f(x)]\ ;\ x\varepsilon G_1\})$ by Fact 1.4.

But, $Der(f[g[B]],f[g[E]]\cup\{[f(g(f(x)))\to f(x)]\ ;x\varepsilon G_1\})$

$$=Der(B,E\cup\{f(x)\to f(x)]\ ;\ x\varepsilon G_1\})=Der(B,E),$$

because $f[g[B]]=B$, $f[g[E]]=E$ and $[f(g(f(x)))\to f(x)]=[f(x)\to f(x)]$.

So, we have $f(z)\varepsilon Der(B,E)$. Hence, we have $z\varepsilon f^{-1}[Der(B,E)]$.

Since z is an arbitrary element of $Der(g[B],g[E]\cup\{[g(f(x))\to x]\ ;\ x\varepsilon G_1\})$, this means

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that (6) holds.

(Q.E.D)

§ 2. Axiomatizations.

Let T be a subset of G, called here "target set" in G. We assume that T is difficult to understand and we want to express T in more easy style by using inferences on G, where axiomatizations are used.

DEFINITION 2.1 (Axiomatizations).

Let A be a subset of G and let D be a subset of G. If T=Der(A,D), then Der(A,D) is called "axiomatization" of T in G.*

DEFINITION 2.2 (Axioms, Logic, Axiom System and Logic System).

If Der(A,D) is an axiomatization of T in G , then elements of A, elements of D, the set A and the set D are called "axioms", "logic", "axiom system" and "logic system", of this axiomatization, respectively.

Note that the logic system D of an axiomatization Der(A,D) is used as logic (correct inferences) in this axiomatization.

FACT 2.1. *We can assume that the logic system D is a subset of T*, without loss of generality, because T=Der(A,D)=Der(A,D∩T*) .*

Note that Der({a},{[a→x] ε G* ; x ε T}) is an axiomatization of T, for each element a of T. These axiomatizations are called "trivial axiomatizations" of T.

Almost all axiomatizations of T are not useful to understand T. So, we would like to consider useful axiomatizations.

In order to define useful axiomatizations, we use the notion "concrete method" without definition because its definition is very complicated, but familiar among us. This notion is used in the form "concrete method to decide something" or "decide something by a concrete method" (cf.[3]).

Also, here after, ground set G and the set G* of inferences on G are all concrete sets, in the following sense:

There is a concrete method to decide x=y or not, for each element x and y of G.

There is a concrete method to decide d=e or not, for each element d and e of G*.

There is a concrete method to obtain all premises and conclusion of d, for each element d of G*.

DEFINITION 2.3 (Decidable Sets and Semi-Decidable Sets).

A subset U of a set V is called "decidable" in V if there is a concrete method to decide whether x belongs to U or not, for any x in V.

A subset U of a set V is called "semi-decidable" in V if there is a set W and a

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decidable subset Z of the direct product $V \times W$ *such that* $U=\{x;\ <x,y>\ \varepsilon\ Z$ *for some* y *in* $W\}$.

Then, we have the following facts:

FACT 2.2. *If U is a decidable in V and V is a decidable in W,*
then U is decidable in W.

FACT 2.3. *If U and V are decidable in W,*
then $U \cup V$ *(union set) is also decidable in W.*

DEFINITION 2.4 (Decidable Mappings).

A mapping f from U to V is called "decidable" if there is a concrete method to obtain the value f(x), for each x in U.

FACT 2.4. *If f is a decidable mapping from U to V and g is a decidable mapping from V to W ,then the composite mapping of f and g is a decidable mapping from U to W.*

DEFINITION 2.5 (Back-Decidable Mappings).

Suppose that f is a mapping from U to V. Then, f is called "back-decidable" if the set $f^{-1}[\{y\}]=\{x\varepsilon U;\ f(x)=y\}$ *is finite and there is a concrete method to obtain a list of all elements of this set, for each y in Y.*

FACT 2.5. *Suppose that f is a back-decidable mapping from* G_1 *to* G_2 *.*
(i) If A is a decidable subset of G_1*, then f[A] is also decidable in* G_2*.*
(ii) If D is a decidable subset of G_1^**, then f[D] is also decidable in* G_2^**.*

(Proof)

Assume that A is a decidable subset of G_1 and f is a back-decidable mapping from G_1 to G_2. Let y be an arbitrary element of G_2 and $x_1, x_2,...,x_n$ be a list of $\{x;f(x)=y\}$.

If there is x_i such that $x_i\ \varepsilon A$, then $y\varepsilon f[A]$.

Otherwise, y does not belong to f [A].

This is a concrete method to decide whether y belongs to f[A] or not, for each y in G_2 . This shows that (i) holds.

Assume that D is a decidable subset of G_1^* and f is a back-decidable mapping from G_1 to G_2. Let d be an arbitrary element of G_2^* and $d= [y_1,y_2,...,y_n \rightarrow y]$. By using lists of the sets $\{x;f(x)=y_1\},\{x;f(x)=y_2\},...,\{x;f(x)=y_n\},\{x;f(x)=y\}$, we have a list of the set $\{e\varepsilon G_1^*\ ;f(e)=d\}$.

If there is e in this list such that $e\varepsilon D$, then $d\varepsilon$ f[D].

Otherwise, d does not belong to f[D].

This is a concrete method to decide whether d belongs to f[D] or not, for each d in G_2^*. This shows that (ii) holds. (Q.E.D.)

FACT 2.6. *Suppose that f is a decidable mapping from G to G. Then*
$\{[f(x)\rightarrow x]\varepsilon G^*\ ;\ x\varepsilon G\}$ *is decidable in* G^*.

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(Proof) Let d be an arbitrary element of G*. Then, we can obtain the premises and the conclusion y of d. If d has only one premise z and f(y) = z, then d ε {[f(x)→x];xεG}. Otherwise d does not belong to {[f(x)→x];xεG}.

(Q.E.D.)

DEFINITION 2.6 (Complete axiomatization).

An axiomatization of T in G is called "complete" if its axiom system is decidable in G and its logic system is decidable in G.*

Suppose that T=Der(A,D) is a complete axiomatization of T. Then,

T=Der(A,D) ={x; there is a D-proof figure h
such that the root of h is x and every top of h belongs to A. }.

On the other hand, the subset

{<x,h>; h is a D-proof figure
such that the root of h is x and every top of h belongs to A.}

is a decidable subset of G×H, where H is the set of D-proof figures, because A is decidable in G and D is decidable in G*.

Hence, we have:

REMARAK 2.1. *If T has a complete axiomatization, then T is semi-decidable.*

Then, we obtain the following two theorems by using Facts 2.3, 2.4, 2.5 and 2.6.

THROREM 2.1 (Forward Axiomatization Theorem).

Suppose that T is a subset of G_1 and f is T-conjugate and back-decidable mapping from G_1 to G_2. If T has a complete axiomatization in G_1, then f[T] has a complete axiomatization in G_2.

(Proof)

Assume that T has a complete axiomatization in G_1. Then, there are a decidable subset A of G_1 and a decidable subset D of G_1* such that T=Der(A,D).

By Theorem 1.2 (Forward Derivation Theorem), we have f[T]=Der(f[A],f[D]) .

Also, by Fact 2.5, f[A] is a decidable subset of G_2 and f[D] is a decidable subset of G_2*. Therefore, f[T] has a complete axiomatization in G_2 .

(Q.E.D.)

THEOREM 2.2 (Backward Axiomatization Theorem).

Let T be a subset of G_2. Suppose that f is a decidable mapping from G_1 to G_2 and g is a decidable mapping form G_2 to G_1 such that f(g(y))=y, for all y in G_2.

If T has a complete axiomatization in G_2, then f^{-1}[T] has a complete axiomatization in G_1.

(Proof)

Assume that T has a complete axiomatization in G_2. Then, there are a decidable

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subset B of G_2 and a decidable subset E of G_2* such that T=Der(B,E).

By Theorem 1.3 (Backward Derivation Theorem), we have

$$f^{-1}[T]=\mathrm{Der}(g[B],g[E]\cup\{[g(f(x))\rightarrow x]\ ;\ x\varepsilon G_1\})\ .$$

Here, we shall show that g is back-decidable.

By the assumption that f(g(y))=y, for all y in G_2, we have:

y ε $g^{-1}[\{x\}]$ implies f(x)=y and x=g(f(x)), for each y in G_2 and each x in G_1.----(*)

Let x be an element of G_1 and let y be an element of G_2 .

Here, decide x=g(f(x)) holds or not by Fact 2.4.

If x=g(f(x)), then $g^{-1}[\{x\}]=\{f(x)\}$ by (*).

If x≠g(f(x)), then $g^{-1}[\{x\}]=\phi$ (the empty set) by (*).

Therefore, g is back-decidable.

By Fact 2.5, g[B] is decidable in G_1 and g[E] is decidable in G_1*.

Moreover, by Fact 2.3 and Fact 2.6, $g[E]\cup\{[g(f(x))\rightarrow x]\ ;x\varepsilon G_1\}$ is decidable in G_1*.

Hence, $\mathrm{Der}(g[B],g[E]\cup\{[g(f(x))\rightarrow x]\ ;x\varepsilon G_1\})$ is a complete axiomatization of $f^{-1}[T]$.

This means that $f^{-1}[T]$ has a complete axiomatization in G_1.

(Q.E.D.)

§ 3. Valuation frames and Gentzen-Hilbert Theorem.

In this section, we shall consider ground sets with valuation frames. By using these valuation frames, we can define true elements of G and correct inferences (logic) in G*. Then, there is a close connection between complete axiomatizations of the set of true elements of G and complete axiomatizations of the set of correct inferences in G*.

DEFINITION 3.1 (Valuation Frame).

A Valuation Frame is a triples <V,G,val>, where V ,G are non-empty sets and val is a mapping from the product set V×G to the set {0,1}.

DEFINITION 3.2 (Valuation Set, Ground set, Valuation Mapping and Valuers).

If <V,G,val> is a valuation frame, then V ,G, val and elements of V are called "valuation set", "ground set", "valuation mapping" and "valuers" of this valuation frame, respectively.

Here, we fix a valuation frame <V,G,val> such that G and G* are concrete sets.

Let v be a valuer and let x be an element of G. Then "val(v,x)=1" is a condition on v and x. In this paper, we use "x is true for v" to denote this condition. Therefore, "val(v,x)=1" is equivalent to "x is true for v ".

Now, we shall define logically true elements of G and logically correct inferences in G*.

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DEFINITION 3.3 (logically true elements of G).

An element x of the ground set G is called "logically true" if x is true for any valuer.

DEFINITION 3.4 (logically correct inferences in G*).

An inference d in G is called "correct" for v if at least one premise of d is not true for v or the conclusion of d is true for v.*

An inference d in G is called "logically correct" if d is correct for any valuer.*

The main purpose of this paper is to investigate relations between "logically true elements" and "logically correct inferences". Let Tt be the set of logically true elements of G and let Tc be the set of logically correct inferences in G*. Then, we can consider Tt as a new target set in G and Tc as a new target set in G*.

DEFINITION 3.5. *A mapping neg from G to G is called "negation mapping" if*

val(v,neg(x))=1-val(v,x) , for each valuer v and each x in G.

DEFINITION 3.6. *A mapping dis form G×G to G is called "disjunction mapping" if*

val(v,dis(x,y))=1-(1-val(v,x)) (1-val(v,y)), for any valuer v and any elements x,y of G.

In the following, we use ¬x to denote negation mapping and x∨y to denote disjunction mapping. Therefore,

val(v,¬x)=1-val(v,x), val(v,x∨y))=1-(1-val(v,x)) (1-val(v,y)).

Then,

" ¬x is true for v" is equivalent to " x is not true for v" ,

"x∨y is true for v" is equivalent to "x is true for v or y is true for v ".

Here, we assume that there are decidable and back-decidable negation mapping "¬" and decidable and back-decidable disjunction mapping "∨".

From negation mapping and disjunction mapping, we define Ker (called "Kernel mapping") and Lif (called "Lifting mapping") as follows:

DEFINITION 3.7. *Ker is a mapping from G* to G defined by*

$$Ker(d)= \neg x_1 \vee \neg x_2 \vee \ldots \vee \neg x_n \vee x, \text{ where } d=[x_1,x_2,\ldots,x_n \to x].$$

DEFINITION 3.8. *Lif is a mapping from G to G* defined by Lif(x)=[→x].*

Then, we have the following important theorem which shows a relation between logically true elements and logically correct inferences:

THEOREM 3.1 (Deduction Theorem).

(i) d is correct for v if and only if Ker(d) is true for v, for any valuer v.

(ii) d is logically correct if and only if Ker(d) is logically true.

From this theorem, we have:

FACT 3.1. *Ker is Tc-conjugate.*

Also, by definitions of Ker and Lif we have:

FACT 3.2. *Ker and Lif are decidable and back-decidable.*

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FACT 3.3. *Ker(Lif(x))=x for all x in G.*

FACT 3.4. *Ker[Tc]=Tt and* Ker^{-1}*[Tt]=Tc*

By Theorem 2.1(Forward Axiomatization Theorem) and Theorem 2.2(Backward Axiomatization Theorem), we have:

THROREM 3.2 (Gentzen-Hilbert Theorem).

Tc has a complete axiomatization in G if and only if Tt has a complete axiomatization in G.*

Two statements "Tc has a complete axiomatization in G*" and "Tt has a complete axiomatization in G" are called "Completeness Theorem of Gentzen Type" and "Completeness Theorem of Hilbert Type", respectively. These two statements are conditions on <V,G,val>. Therefore, Gentzen-Hilbert Theorem above shows that Completeness Theorem of Gentzen Type and Completeness Theorem of Hilbert Type are equivalent as conditions on <V,G,val>, which satisfy some necessary requirements.

§ 4. Formal Languages and Valuation Frames.

In this section, we shall consider formal languages and their valuation frames. We assume that the readers are familiar with formal languages. (cf. [1])

DEFINITION 4.1 (Formal Languages).

A formal language L is defined by the following three rules; rule of symbols, rule of formations and rule of interpretations.

(1) Rule of symbols*: Rule of symbols define the scope of symbols we can use in L. Such symbols are called "admitted symbols" in L. Note that admitted symbols correspond words in our ordinary languages.*

(2) Rule of formations*: Rule of formations define the scope of sequences of admitted symbols, we can use in L. Such sequences are called "admitted sequences" in L. Note that admitted sequences correspond to phrases or sentences in our ordinary languages. Some admitted sequences are called "formal sentences" which are intended to express information.*

(3) Rule of interpretations*: Rule of interpretations are rules to assign meanings to each admitted symbols in L, by using tools called "L-structures".*

We assume that rules of symbols and rules of formations here are concrete rule. Therefore, formal sentences are concrete objects and inferences between formal sentences are concrete objects. On the other hand, L-structures are not concrete objects and rules of interpretations are not concrete rules.

Let M be an L-structure and let F be a formal sentence. Using rule of

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interpretations, we can obtain a sentence by assigning meanings to all symbols in F by M. Such sentence is denoted by F[M]. If the sentence F[M] is true, we say that "F is true for M".

To understand formal languages, we shall explain admitted symbols more.

Admitted symbols of L is divided into two groups. One is variables and the other is constants.

Variables are classified by their intended domains which depend on L-structures. For examples, domains of individual variables consist of individuals in L-structures and domains of set variables consist of sets of individuals in L-structures.

Constants are divided into two groups. One is constants whose meanings depend on L-structures. For example, meanings of individual constants, function constants, predicate constants, set constants depend on L-structures.

The other is constants whose meanings are independent of L-structures. Such symbols are called "logical symbols".

As logical symbols we use "=" (equality), "¬" (negation), "∧" (conjunction) ,"∨" (disjunction), "∀" (universal quantifier) and "∃" (existential quantifier), etc.

Of course, meanings of logical symbols are clear for us.

Here, we introduce a valuation frame <St(L),Fs(L),sat> whose valuation set is the class St(L) of L-structures ,whose ground set is the set Fs(L) of formal sentences in L and whose valuation mapping is the mapping sat from the product class St(L)× Fs(L) to the set {0,1},such that sat(M,F)=1 if F is true for M and sat(M,F)=0 otherwise, for each L-structure M and each formal sentence F .

Note that St(L) is a proper class in the sense of Naive Set Theory. But, we can replace St(L) by its subset by Lowenheim-Skolem Theorem without loss of generality. So, we consider St(L) as a set in the following of this paper.

Note that the ground set Fs(L) and the set Fs(L)* of inferences are concrete sets.

Then, by Definitions 3.3 and 3.4, we can define "logically true sentences" in Fs(L) and "logically correct inferences" in Fs(L)*.

DEFINITION 4.2 (Logically true sentences).

Let F be a formal sentence in Fs(L) and M is an L-structure in St(L).

F is true for M if the sentence F[M] is true.

F is logically true if F is true for any L-structure in St(L).

Let Tt(L) be the set of logically true sentences in L.

DEFINITION 4.3 (Logically correct inferences).

Let d be an inference in Fs(L) and M is an L-structure in St(L).*

d is correct for M if at least one premise of d is not true for M or the conclusion of d

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is true for M.

d is logically correct if d is correct for any L-structure M in St(L).

Let Tc(L) be the set of logically correct inferences in Fs(L).*

Next, we introduce the negation mapping "neg" and disjunction mapping "dis" which are defined by negation symbol and disjunction symbol as follows:

neg(F)=¬F and dis(F,G)=F∨G for each formal sentences F and G.

Note that "neg" and "dis" defined here, are examples of negation mapping and disjunction mapping introduced in §3, respectively.

By interpretation rule of L,

¬F if true for M if and only if F is not true for M,

F∨G is true for M if and only if F is true for M or G is true for M.

Since ¬F ,F∨G are concrete sequences, the negation mapping and disjunction mapping are decidable and back-decidable.

As in §3, we can define Kernel mapping Ker and Lifting mapping Lif as follows:

$Ker(d)=\neg F_1\vee\neg F_2\vee\ldots\vee\neg F_n \vee F$, where $d=[F_1,F_2,\ldots,F_n\rightarrow F]$,

$Lif(F)=[\ \rightarrow F]$.

Then, Ker and Lif are decidable and back-decidable. Also, these mappings satisfy all conditions stated in § 3. Therefore, we have the following theorem:

THEOREM 4.1 (Gentzen-Hilbert Theorem of L).

Tc(L) has a complete axiomatization in Fs(L) if and only if Tt(L) has a complete axiomatization in Fs(L).*

In case that L is a first-order language, Tc(L) has a complete axiomatization in Fs(L)* and Tt(L) has a complete axiomatization in Fs(L). The former is called "Godel Completeness of Gentzen type" and the latter is called "Godel Completeness of Hilbert type".

Let W be a subclass of St(L). Then, we can introduce a valuation frame <W,Fs(L),sat_W> whose valuation set is W, whose ground set is Fs(L) and whose valuation mapping is the mapping sat_W obtained from sat by restricting its domain to W×Fs(L).

Here, we can define "logically true sentences" in Fs(L) and "logically correct inferences" in Fs(L)* with respect to the valuation frame <W,Fs(L),sat_W> .

Then, there are two kinds of "logically true sentences" and "logically correct inferences". One is those with respect to <St(L),Fs(L),sat> and another is those with respect to <W,Fs(L),sat_W>.

In order to distinguish between them, we use "W-true sentences" and "W-correct inferences", instead of "logically true sentences" and "logically correct inferences" with respect to <W,Fs(L),sat_W>.

Therefore, for each sentence F in L and each inference d in L*,

F is W-true sentence if and only if F is true for any L-structure in W,

d is W-correct inference if and only if d is correct for any L-structure in W.

Let Tt(W) be the set of W-true sentences in L and Tc(W) be the set of W-correct inferences in L.*

Then, Deduction Theorem of the form “d is W-correct inference if and only if Ker(d) is W-true sentence” holds.

Hence we have that Ker[Tc(W)]=Tt(W) and Ker^{-1}[Tt(W)]=Tc(W).

Therefore, we obtain the following generalization of Genzten-Hilbert Theorem.

THEOREM 4.4 (Generalized Gentzen-Hilbert Theorem).

Tc(W) has a complete axiomatization in Fs(L) if and only if Tt(W) has a complete axiomatization in Fs(L).*

Let W be St(L) in the above theorem, then we have Theorem 4.1(Gentzen-Hilbert Theorem).

Finally, we would like to comment that formal languages treated here include not only first-order languages but also many other formal languages.

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Mathematics Subject Classification 03B22

Key Words and Phrases: Deduction Theorem, Completeness Theorem